JOURNAL OF COMPUTATIONAL PHYSICS 3, 80–93 (1968)

Numerical Calculation of Almost Incompressible Flow

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Received March 25, 1968

ABSTRACT

A new method is presented for the numerical solution of time-dependent problems in several space dimensions. The technique is applicable to low-speed (incompressible) flows, to high-speed (supersonic) flows, and to all flow speeds in between, thereby bridging the gap through the almost-incompressible regime in which previously-described techniques break down.

INTRODUCTION

Most numerical methods for the solution of transient problems in fluid dynamics are applicable to only a restricted range of fluid speeds. For sonic or faster flows there are Lagrangian [1], Eulerian [2] and hybrid [3] techniques that have proved successful for numerous applications. For completely incompressible flows there are also several useful methods, all Eulerian [4], [5].

To bridge the gap, we have developed an Implicit Continuous-Fluid Eulerian (ICE) method that has proved successful for flow calculations in all velocity ranges. In the incompressible limit, the technique reduces to the MAC method, [5] while for supersonic flows it is an implicit variant of the usual Eulerian procedures. For all flow speeds (Mach numbers from zero to infinity) the ICE method gives a numerically stable and efficient means for calculating transient, viscous fluid flows in several space dimensions. It also serves as a basic technique in which visco-plastic effects can be included.

The ICE procedure is based on an implicit finite-difference approximation to the full non-linear equations of fluid dynamics. Starting from prescribed initial conditions, the solution advances through a sequence of cycles, each step representing the configuration at a small but finite time interval, δt, later than the previous one. Space derivatives are represented by appropriate finite-difference approximations that are related to an Eulerian mesh of cells. The calculation sequence for each cycle is as follows.

1. The density for each cell is found by iterative solution of a finite-difference Poisson's equation. Efficiency is increased in this solution by utilizing a previously-described corrective procedure [6].

2. The pressures for all the cells are calculated, using whatever equation of state is appropriate for the fluid.

3. The velocities for all cells are advanced in time by means of finite-difference momentum equations.

4. The new specific total energies and specific internal energies are calculated for all the cells.

5. Lagrangian marker particles are moved, in order to denote the change in fluid configuration. (In confined flow problems these particles are optional as they do not enter into the dynamics; their principal utility arises in multi-fluid or free surface calculations, in which they mark the interface positions that are essential for applying boundary conditions.)

THE EQUATIONS

We illustrate the ICE technique for a material described by the equation of state

$$p = a^2(\rho - \rho_0) + (\gamma - 1)\,\rho I, \tag{1}$$

in which the pressure, p, relates to the specific internal energy I, and the density, ρ. "Normal" density is the constant, ρ_0 ; γ is a numerical constant slightly greater than unity; and a is the sound speed at normal density and zero specific internal energy. An incompressible fluid is represented by $a^2 \to \infty$.

For illustration, we limit the viscosity to a simple "artificial" form, rather than incorporating the full stress tensor. Some viscosity is shown to be necessary for the calculation of shock dynamics and other high-speed flows. The illustration is also restricted to plane two dimensional flows, for which we start from the equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{2}$$

$$\frac{\partial \rho u}{\partial t} + \frac{\partial \rho u^2}{\partial x} + \frac{\partial \rho uv}{\partial y} + \frac{\partial P}{\partial x} = 0, \tag{3}$$

$$\frac{\partial \rho v}{\partial t} + \frac{\partial \rho uv}{\partial x} + \frac{\partial \rho v^2}{\partial y} + \frac{\partial P}{\partial y} = 0, \tag{4}$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial \rho u E}{\partial x} + \frac{\partial \rho v E}{\partial y} + \frac{\partial P u}{\partial x} + \frac{\partial P v}{\partial y} = 0. \tag{5}$$

The velocities, u and v, are, respectively, components along the coordinate axes x and y. $E \equiv I + \frac{1}{2}(u^2 + v^2)$ is the specific total energy, while $P \equiv p + q$, where q is the artificial viscous pressure. Typically we might have q proportional to the velocity divergence.

The crucial feature of the ICE procedure is the choice of finite-difference representation by which these equations are to be approximated. We have used the following:

$$\frac{\rho_{ij}^{n+1} - \rho_{ij}^{n}}{\delta t} + \frac{(\rho u)_{i+1/2,j}^{n+1} - (\rho u)_{i-1/2,j}^{n+1}}{\delta x} + \frac{(\rho v)_{i,j+1/2}^{n+1} - (\rho v)_{i,j-1/2}^{n+1}}{\delta y} = 0, \tag{6}$$

$$\begin{aligned}\frac{(\rho u)_{i+1/2,j}^{n+1} - (\rho u)_{i+1/2,j}^{n}}{\delta t} &+ \frac{1}{\delta x}\,[\rho_{i+1,j}^{n+1}(u^2)_{i+1,j}^{n} - \rho_{ij}^{n+1}(u^2)_{ij}^{n} \\ &+ \bar{p}_{i+1,j} - \bar{p}_{ij} + q_{i+1,j}^{n} - q_{ij}^{n}] \\ &+ \frac{1}{\delta y}\,[\rho_{i+1/2,j+1/2}^{n+1}(uv)_{i+1/2,j+1/2}^{n} - \rho_{i+1/2,j-1/2}^{n+1}(uv)_{i+1/2,j-1/2}^{n}] = 0,\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{(\rho v)_{i,j+1/2}^{n+1} - (\rho v)_{i,j+1/2}^{n}}{\delta t} &+ \frac{1}{\delta x}\,[\rho_{i+1/2,j+1/2}^{n+1}(uv)_{i+1/2,j+1/2}^{n} - \rho_{i-1/2,j+1/2}^{n+1}(uv)_{i-1/2,j+1/2}^{n}] \\ &+ \frac{1}{\delta y}\,[\rho_{i,j+1}^{n+1}(v^2)_{i,j+1}^{n} - \rho_{ij}^{n+1}(v^2)_{ij}^{n} + \bar{p}_{i,j+1} - \bar{p}_{ij} + q_{i,j+1}^{n} - q_{ij}^{n}] = 0,\end{aligned} \tag{8}$$

$$\begin{aligned}\frac{(\rho E)_{ij}^{n+1} - (\rho E)_{ij}^{n}}{\delta t} &+ \frac{1}{\delta x}\,[(\rho u)_{i+1/2,j}^{n+1} E_{i+1/2,j}^{n} - (\rho u)_{i-1/2,j}^{n+1} E_{i-1/2,j}^{n} \\ &+ (\bar{p}_{i+1/2,j} + q_{i+1/2,j}^{n})\, u_{i+1/2,j}^{n+1} - (\bar{p}_{i-1/2,j} + q_{i-1/2,j}^{n})\, u_{i-1/2,j}^{n+1}] \\ &+ \frac{1}{\delta y}\,[(\rho v)_{i,j+1/2}^{n+1} E_{i,j+1/2}^{n} - (\rho v)_{i,j-1/2}^{n+1} E_{i,j-1/2}^{n} \\ &+ (\bar{p}_{i,j+1/2} + q_{i,j+1/2}^{n})\, v_{i,j+1/2}^{n+1} - (\bar{p}_{i,j-1/2} + q_{i,j-1/2}^{n})\, v_{i,j-1/2}^{n+1}] = 0,\end{aligned} \tag{9}$$

where

$$\bar{p} \equiv a^2(\rho^{n+1} - \rho_0) + (\gamma - 1)\,\rho^{n+1} I^{n}.$$

The various indices are related to the mesh of cells and to the finite-difference time advancement; the indices i and j count cell-center positions in the x and y directions, respectively, while $i \pm \frac{1}{2}$ and $j \pm \frac{1}{2}$ are cell boundary positions. The superscript index, n, counts time cycles. The specific quantities, ρ, E and p, are defined at cell centers; thus cell-boundary values are obtained as simple averages of the two adjacent quantities while cell-corner values are the averages of four adjacent quantities. The velocities are defined at cell boundaries with u at the $i \pm \frac{1}{2}$ points and v at the $j \pm \frac{1}{2}$ points. Again, cell-center or cell-corner velocities are obtained as averages. These centerings are illustrated in Fig. 1.

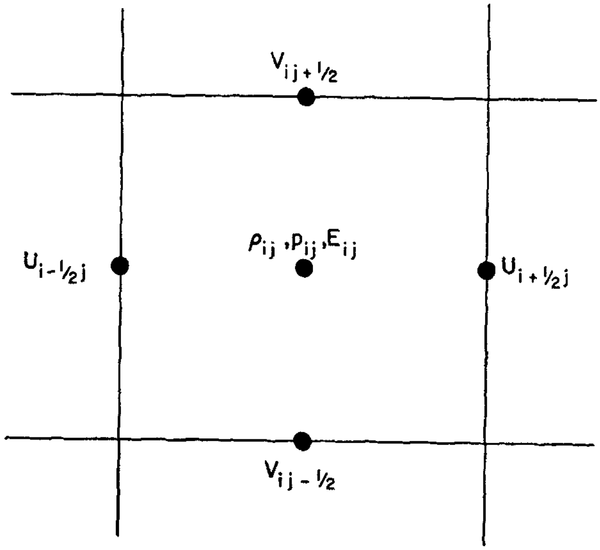

FIG. 1. Placement of field variables about an ICE cell. Density, pressure, and energy are defined at the cell center, while velocity components are defined at the cell boundaries normal to the components.

METHOD OF SOLUTION

The essential features of the ICE technique are contained in the finite-difference equations, and particularly in the time centering of the various terms, as will be demonstrated by analysis of the truncation errors and the limiting cases of slow and fast flow. The procedure for solving these equations, however, introduces several features that are also of importance to the success of ICE.

Foremost among these secondary features is the introduction of the auxiliary function

$$\sigma \equiv a^2(\rho - \rho_0) + K\rho \tag{10}$$

in which K is a constant. The general prescription for defining σ for any equation of state will become clear as soon as the purpose of σ has been demonstrated.

To see this, we eliminate the advanced-time velocities from Eqs. (6), (7) and (8). First, from Eq. (6) is subtracted the same equation in which the index n has everywhere been reduced by one. Then, from Eq. (7) is subtracted the same equation in which the index i has everywhere been reduced by one, while likewise for Eq. (8) one subtracts the same equation with j reduced by one. The three results can then be combined in such a way as to eliminate all of the advanced-time velocities, the result being

$$\begin{aligned}
\frac{\rho_{ij}^{n+1} + \rho_{ij}^{n-1} - 2\rho_{ij}^{n}}{\delta t^2} = {} & \frac{1}{\delta x^2}\,[\rho_{i+1,j}^{n+1}(u^2)_{i+1,j}^{n} - 2\rho_{ij}^{n+1}(u^2)_{ij}^{n} + \rho_{i-1,j}^{n+1}(u^2)_{i-1,j}^{n} \\
& + \bar{p}_{i+1,j} - 2\bar{p}_{ij} + \bar{p}_{i-1,j} + q_{i+1,j}^{n} - 2q_{ij}^{n} + q_{i-1,j}^{n}] \\
& + \frac{1}{\delta y^2}\,[\rho_{i,j+1}^{n+1}(v^2)_{i,j+1}^{n} - 2\rho_{ij}^{n+1}(v^2)_{ij}^{n} + \rho_{i,j-1}^{n+1}(v^2)_{i,j-1}^{n} \\
& + \bar{p}_{i,j+1} - 2\bar{p}_{ij} + \bar{p}_{i,j-1} + q_{i,j+1}^{n} - 2q_{ij}^{n} + q_{i,j-1}^{n}] \\
& + \frac{2}{\delta x \delta y}\,[\rho_{i+1/2,j+1/2}^{n+1}(uv)_{i+1/2,j+1/2}^{n} - \rho_{i+1/2,j-1/2}^{n+1}(uv)_{i+1/2,j-1/2}^{n} \\
& - \rho_{i-1/2,j+1/2}^{n+1}(uv)_{i-1/2,j+1/2}^{n} + \rho_{i-1/2,j-1/2}^{n+1}(uv)_{i-1/2,j-1/2}^{n}]. \qquad (11)
\end{aligned}$$

This is the principal equation to be solved. The unknown quantities are the various new-time densities, ρ^{n+1}, of which nine different ones appear in the equation, corresponding to the cell i, j, and its eight neighbors.

To solve the coupled set of equations given by Eq. (11), it is convenient to use some method of iteration, cycling through the mesh until the new density field satisfies Eq. (11) everywhere to some prescribed degree of accuracy. For high-speed flows, in which the density varies appreciably from ρ_0, the iteration for new densities can be accomplished with the equation as it is. For low-speed flows, however, the density scarcely differs from ρ_0 and accordingly is unsatisfactory as an iteration variable. This, then, is the reason for introducing σ; with σ as an iteration variable and K chosen to equal the square of some representative fluid speed, the iteration on σ will be satisfactory for all possible values of a^2/K from zero to very large. Indeed, as $a \to \infty$, the σ-variable form of Eq. (11) [Eq. (12), below] then reduces directly to the iterative equation for pressure used in the MAC method [5]. [Our preliminary ICE technique proof-test calculations used Eq. (11) successfully for Mach numbers down to about 0.2; below that, the loss of significance in density variations required inefficient iteration convergence restrictions. A change to iteration on σ cured the difficulty completely.]

Inserting Eq. (10) into Eq. (11), we obtain

$$
\begin{aligned}
&\sigma_{ij}^{n+1} + \sigma_{ij}^{n-1} - 2\sigma_{ij}^{n} \\
&\quad = \frac{\delta t^2}{\delta x^2}\{[\sigma_{i+1,j}^{n+1} + a^2\rho_0][(u^2)_{i+1,j}^{n} + (\gamma - 1)\, I_{i+1,j}^{n} + a^2] \\
&\qquad + [\sigma_{i-1,j}^{n+1} + a^2\rho_0][(u^2)_{i-1,j}^{n} + (\gamma - 1)\, I_{i-1,j}^{n} + a^2] \\
&\qquad - 2[\sigma_{ij}^{n+1} + a^2\rho_0][(u^2)_{ij}^{n} + (\gamma - 1)\, I_{ij}^{n} + a^2] \\
&\qquad + (q_{i+1,j}^{n} + q_{i-1,j}^{n} - 2q_{ij}^{n})(a^2 + K)\} \\
&\qquad + \frac{\delta t^2}{\delta y^2}\{[\sigma_{i,j+1}^{n+1} + a^2\rho_0][(v^2)_{i,j+1}^{n} + (\gamma - 1)\, I_{i,j+1}^{n} + a^2] \\
&\qquad + [\sigma_{i,j-1}^{n+1} + a^2\rho_0][(v^2)_{i,j-1}^{n} + (\gamma - 1)\, I_{i,j-1}^{n} + a^2] \\
&\qquad - 2[\sigma_{ij}^{n+1} + a^2\rho_0][(v^2)_{ij}^{n} + (\gamma - 1)\, I_{ij}^{n} + a^2] \\
&\qquad + (q_{i,j+1}^{n} + q_{i,j-1}^{n} - 2q_{ij}^{n})(a^2 + K)\} \\
&\qquad + \frac{\delta t^2}{2\delta x \delta y}\{[\sigma_{i+1,j+1}^{n+1} + \sigma_{i,j+1}^{n+1} + \sigma_{i+1,j}^{n+1} + \sigma_{ij}^{n+1} + 4a^2\rho_0][(uv)_{i+1/2,j+1/2}^{n}] \\
&\qquad - [\sigma_{i+1,j}^{n+1} + \sigma_{ij}^{n+1} + \sigma_{i+1,j-1}^{n+1} + \sigma_{i,j-1}^{n+1} + 4a^2\rho_0][(uv)_{i+1/2,j-1/2}^{n}] \\
&\qquad - [\sigma_{i,j+1}^{n+1} + \sigma_{ij}^{n+1} + \sigma_{i-1,j+1}^{n+} + \sigma_{i-1,j}^{n+1} + 4a^2\rho_0][(uv)_{i-1/2,j+1/2}^{n}] \\
&\qquad + [\sigma_{ij}^{n+1} + \sigma_{i,j-1}^{n+1} + \sigma_{i-1,j}^{n+1} + \sigma_{i-1,j-1}^{n+1} + 4a^2\rho_0][(uv)_{i-1/2,j-1/2}^{n}]\}. \qquad (12)
\end{aligned}
$$

We next define

$$A_{ij} \equiv \frac{\delta t^2}{\delta x^2}\,[(u^2)_{ij}^{n} + a^2 + (\gamma - 1)\, I_{ij}^{n}], \qquad (13)$$

$$B_{ij} \equiv \frac{\delta t^2}{\delta y^2}\,[(v^2)_{ij}^{n} + a^2 + (\gamma - 1)\, I_{ij}^{n}], \qquad (14)$$

$$C_{i+1/2,j} \equiv \frac{\delta t^2}{2\delta x \delta y}\,[(uv)_{i+1/2,j+1/2}^{n} - (uv)_{i+1/2,j-1/2}^{n}], \qquad (15)$$

$$D_{i,j+1/2} \equiv \frac{\delta t^2}{2\delta x \delta y}\,[(uv)_{i+1/2,j+1/2}^{n} - (uv)_{i-1/2,j+1/2}^{n}], \qquad (16)$$

$$F_{i+1/2,j+1/2} \equiv \frac{\delta t^2}{2\delta x \delta y}\,(uv)_{i+1/2,j+1/2}^{n}\,, \qquad (17)$$

$$
\begin{aligned}
G_{ij} \equiv{}& \sigma_{ij}^{n-1} - 2\sigma_{ij}^{n} - a^2\rho_0(A_{i+1,j} + A_{i-1,j} - 2A_{ij} + B_{i,j+1} + B_{i,j-1} - 2B_{ij}) \\
&- (a^2 + K)\left[\frac{\delta t^2}{\delta x^2}(q_{i+1,j}^{n} + q_{i-1,j}^{n} - 2q_{ij}^{n}) + \frac{\delta t^2}{\delta y^2}(q_{i,j+1}^{n} + q_{i,j-1}^{n} - 2q_{ij}^{n})\right] \\
&- 4a^2\rho_0(C_{i+1/2,j} - C_{i-1/2,j}). \qquad (18)
\end{aligned}
$$

Then Eq. (12) can be written

$$
\begin{aligned}
&- F_{i-1/2,j-1/2}\sigma_{i-1,j-1}^{n+1} + (D_{i,j-1/2} - B_{i,j-1})\,\sigma_{i,j-1}^{n+1} + F_{i+1/2,j-1/2}\sigma_{i+1,j-1}^{n+1} \\
&+ (C_{i-1/2,j} - A_{i-1,j})\,\sigma_{i-1,j}^{n+1} + (1 + 2A_{ij} + 2B_{ij} - C_{i+1/2,j} - C_{i-1/2,j})\,\sigma_{ij}^{n+1} \\
&- (C_{i+1/2,j} + A_{i+1,j})\,\sigma_{i+1,j}^{n+1} + F_{i-1/2,j+1/2}\sigma_{i-1,j+1}^{n+1} - (D_{i,j+1/2} + B_{i,j+1})\,\sigma_{i,j+1}^{n+1} \\
&- F_{i+1/2,j+1/2}\sigma_{i+1,j+1}^{n+1} + G_{ij} = 0. \qquad (19)
\end{aligned}
$$

This is the equation for σ^{n+1} to be solved. Because of the advanced-time values, σ^{n+1}, appearing in the momentum convection terms, this is a nine-point equation; it could be reduced to a five-point equation through the use of σ^n in those convection terms, but it can be shown that this would have a small adverse affect on numerical stability. Solution of Eq. (19) can be accomplished by any of several existing techniques. For example, a convenient over-relaxation process of iteration is accomplished by the formula

$$
\begin{aligned}
\bar{\sigma}_{ij}^{n+1} = {}& -\alpha\sigma_{ij}^{n+1} + \frac{1+\alpha}{1 + 2A_{ij} + 2B_{ij} - C_{i+1/2,j} - C_{i-1/2,j}}\,[F_{i-1/2,j-1/2}\bar{\sigma}_{i-1,j-1}^{n+1} \\
&- (D_{i,j-1/2} - B_{i,j-1})\,\bar{\sigma}_{i,j-1}^{n+1} + F_{i+1/2,j-1/2}\bar{\sigma}_{i+1,j-1}^{n+1} \\
&- (C_{i-1/2,j} - A_{i-1,j})\,\bar{\sigma}_{i-1,j}^{n+1} + (C_{i+1/2,j} + A_{i+1,j})\,\sigma_{i+1,j}^{n+1} + F_{i-1/2,j+1/2}\sigma_{i-1,j+1}^{n+1} \\
&+ (D_{i,j+1/2} + B_{i,j+1})\,\sigma_{i,j+1}^{n+1} + F_{i+1/2,j+1/2}\sigma_{i+1,j+1}^{n+1} - G_{ij}], \qquad (20)
\end{aligned}
$$

in which $\bar{\sigma}$ indicates the new value for the particular iteration, and cycling is implied in the directions of increasing i and j. The overrelaxation parameter, α, is a number slightly exceeding unity, and iteration is repeated until some criterion of convergence is satisfied.

When the new values of σ have been obtained, the densities can be found from Eq. (10), while to find the pressure we use

$$p = \sigma + \rho[(\gamma - 1)\,I - K] \qquad (21)$$

rather than Eq. (1), in order to preserve accuracy in low-speed flows. One then finds, in order, the new velocities, energies and internal energies, using, Eqs. (7)–(9).

Initial and Boundary Conditions

The initial state is described by an arbitrary prescription for the distribution of velocities, density and internal energy, together with the field variables derived from these. In addition, it is necessary to specify the density distribution as it would have appeared at one cycle *before* the starting time. In many three-time-level techniques, this pre-starting-time distribution is also arbitrary, and there is reason for concern as to how the choice should be made. For the ICE method, this difficulty does not arise. In terms of densities and velocities arbitrarily prescribed for the initial state, the pre-starting-time density distribution is determined uniquely by Eq. (6). Indeed, any other choice would lead to fallacious results in the subsequent cycles of the calculation.

Various boundary conditions can be used, and, as in most finite difference computing techniques, some of them can be determined uniquely in terms of the required fluxes of mass, momentum and energy. At a rigid wall, for example, the vanishing of all convective fluxes implies the vanishing of the normal velocity component. If the wall is no-slip, then the tangential velocity also vanishes. At a free surface, the normal fluxes of both normal and tangential momentum must vanish, imposing conditions much like those in MAC [5]. Input walls are easily described by the fluxes that are required, while output boundaries suffer from the same arbitrariness experienced in MAC, and can be treated by the same procedures. As in MAC, it is, in general, essential that the boundary conditions be consistent throughout the equations, in order that Eq. (6), which nowhere is explicitly used in the calculation, is nevertheless accurately satisfied everywhere.

Stability and Accuracy

Hirt [7] has given an effective technique for stability and accuracy analysis that accounts for variability of the coefficients in the finite difference equations. He argues that the principal finite-difference effect in the transport equation for some quantity can be found in the diffusion coefficient for that quantity. In the present case, it is sufficient to consider the one-dimensional versions of Eqs. (6) and (7), which, to lowest contributing order in the diffusion terms, can be written

$$\frac{\partial \rho}{\delta t} + \frac{\partial \rho u}{\delta x} = \left[(c^2 + u^2)\frac{\delta t}{2} - \frac{\delta x^2}{4}\frac{\partial u}{\partial x}\right]\frac{\partial^2 \rho}{\partial x^2} \tag{22}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \left[(\beta c^2 - \epsilon u^2)\frac{\delta t}{2} - \frac{u\delta x^2}{2\rho}\frac{\partial \rho}{\partial x} + \frac{\mu}{\rho}\right]\frac{\partial^2 u}{\partial x^2} \tag{23}$$

in which c is the sound speed, and we have taken

$$q = -\mu\frac{\partial u}{\partial x}. \tag{24}$$

For the ICE-method equations in the proposed form, $\beta = \epsilon = 1$. If the convective terms in the momentum equations contain the ρ^n values (instead of ρ^{n+1}), then Eq. (19) becomes a five-point equation to solve, but $\epsilon = 3$ so that the diffusion coefficient in Eq. (23) is decreased. The result is poorer stability, as mentioned previously.

If the pressure terms were formed with ρ^n instead of ρ^{n+1} in the momentum equations, then $\beta = -1$, and the tendency to instability becomes intolerable as the sound speed increases. This is not unexpected, however, since the presence of ρ^n in the pressure terms renders the momentum equations purely explicit, and these are known to suffer in stagnation regions.

With the proposed form of the ICE method, both the mass and momentum equations exhibit greater and greater diffusion as the sound speed increases. The question arises as to how it is that the limit of infinite sound speed (the MAC method) can give useful results with infinite diffusion coefficients. The answer lies in the fact that for such "incompressible" flows, density does indeed remain constant, as if diffusing with infinite speed, while the infinite diffusion of momentum occurs only in the one-dimensional equations, again reflecting incompressibility.

A consequence of this enhanced diffusion, however, is seen in the widening of shock-front thicknesses as the sound speed becomes large. Our one-dimensional tests show this especially well. Although this causes no concern in two dimensions, one still may inquire into the possibility of a curative modification. Immediately apparent is the effect of using $\frac{1}{2}(\rho^n + \rho^{n+1})$ in the pressure, in which case $\beta = 0$ and the excessive diffusion vanishes. Stability then depends entirely upon the presence of the viscosity term, with the sharpness of the momentum fronts depending upon the value of μ. The corresponding term in Eq. (22) is likewise removed by such a time centering of the mass equation. We have not yet experimented with such modifications but envision the possibility that they could significantly enhance the accuracy. It should be observed, however, that Eq. (22) demonstrates the necessity for inclusion of an artificial mass diffusion term when such time centering is used. Thus, the alternative proposal for the ICE-method equations is of the following form:

$$\frac{\rho_{ij}^{n+1} - \rho_{ij}^{n}}{\delta t} + \frac{(\rho u)_{i+1/2,j}^{n+1/2} - (\rho u)_{i-1/2,j}^{n+1/2}}{\delta x} + \frac{(\rho v)_{i,j+1/2}^{n+1/2} - (\rho v)_{i,j-1/2}^{n+1/2}}{\delta y}$$
$$= \kappa \left[\frac{1}{\delta x^2}(\rho_{i+1,j}^n + \rho_{i-1,j}^n - 2\rho_{ij}^n) + \frac{1}{\delta y^2}(\rho_{i,j+1}^n + \rho_{i,j-1}^n - 2\rho_{ij}^n)\right], \tag{25}$$

together with a redefinition of $\bar{p}$ in Eqs. (7) and (8) such that

$$\bar{p} \equiv a^2(\rho^{n+1/2} - \rho_0) + (\gamma - 1)\,\rho^{n+1/2} I^n.$$

The mass diffusion is controlled by the constant coefficient, κ, and the half-time centering for any quantity is defined

$$(\ )^{n+1/2} \equiv \tfrac{1}{2}[(\ )^n + (\ )^{n+1}].$$

The principal difficulty introduced by this alternative time centering is that of finding the pre-starting-time values of density, velocity and specific internal energy. No longer would Eq. (6) alone be sufficient; it would be necessary to solve simultaneously the entire set of equations, requiring some technique that has not yet been worked out.

There is, however, a second way in which the excess numerical diffusion could be removed from the ICE method. To the mass and momentum equations could be added explicitly diffusion terms with *negative* coefficients exactly (or nearly) cancelling the adverse terms in the error analysis. Thus, for example, Eq. (2) would be replaced, in one space dimension, by

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{1}{2} c^2 \delta t \frac{\partial \rho}{\partial x}\right) \tag{26}$$

thereby removing from Eq. (22) the most troublesome of the excessive diffusion terms.

TABLE I

PARAMETERS FOR THE TWO ONE-DIMENSIONAL ICE-METHOD CALCULATIONS SHOWN IN FIGS. 2 AND 3[a]

Figure No.	2	3
Incoming Mach No.	∞	0.20
No. of Cells	15	50
δx	0.10	
δt	0.01	0.02
$\rho_s = \rho_0$	1.00	
u_s	1.00	
I_s	0	
γ	5/3	
a	0	5.00
μ	0.20	0
K	1.00	

[a] The subscript "s" denotes initial or starting conditions. These same values are maintained at the left (input) boundary during the entire run. The right boundary is a rigid wall.

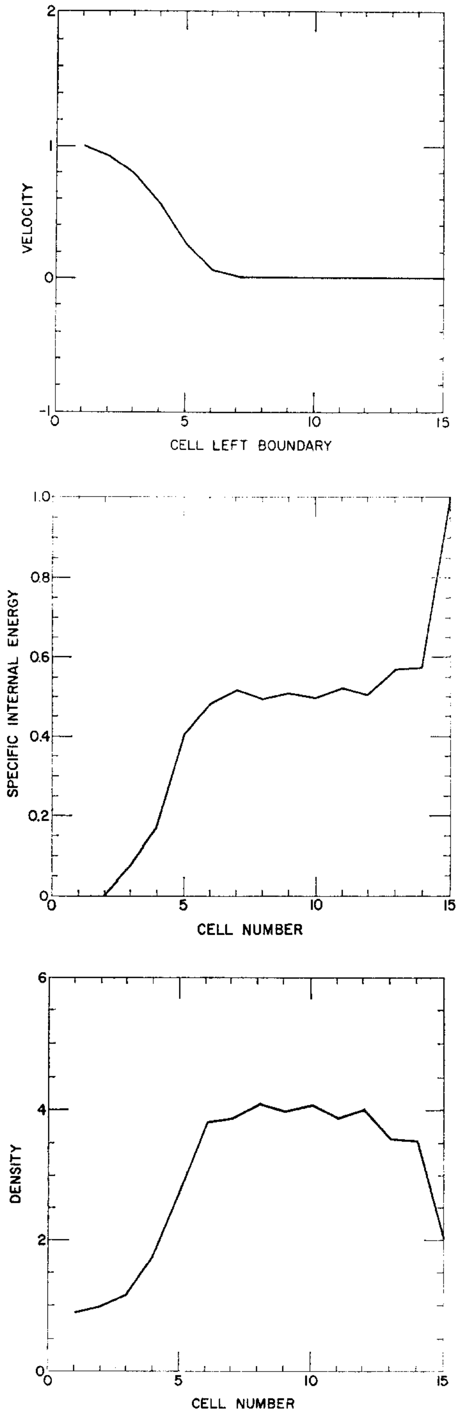

FIG. 2. One-dimensional ICE-method calculation of an infinite-strength shock (incoming Mach number $= \infty$) at a time $t = 3.00$.

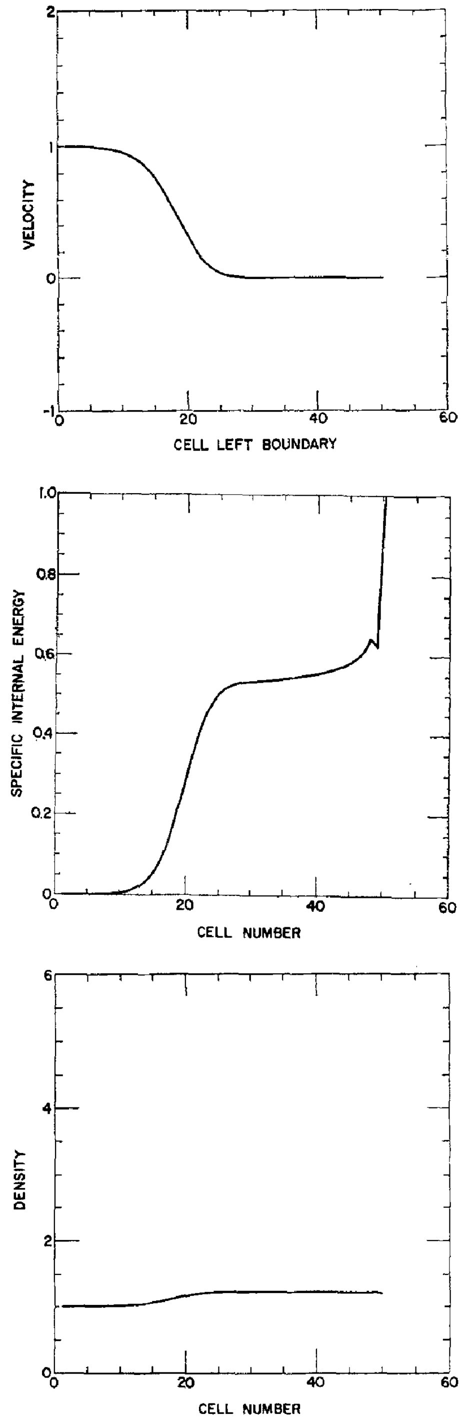

FIG. 3. One-dimensional ICE-method calculation in the far-subsonic incompressible flow regime (incoming Mach number = 0.20) at a time t = 0.70.

Some Calculated Examples

The ICE-method properties are illustrated by the calculation of fluid piling up against a wall. The results, which depend upon only one space variable, are plotted in Figs. 2 and 3 for the representative problems described in Table I.

Figure 2 shows the one-dimensional calculation of an infinite strength shock (incoming Mach number $= \infty$) at a time $t = 3.00$. The three plots are of $u : x$, $I : x$, and $\rho : x$, respectively. The resulting rarefaction wave shows $I_{-}/I_{+} = 0.50$, $\rho_{-}/\rho_{+} = 4.00$, and $p = 4/3$, which agree with simple shock theory. Note that the velocity profile shows no tendency to the usual instabilities of a perturbed stagnation. Note also that the entropy is high at the right (rigid) wall, as is common in Eulerian schemes. This is manifested in a high I and low ρ at the wall; however, p comes to the proper value.

Figure 3 shows the same calculation performed in the far subsonic incompressible flow regime at a time $t = 0.70$. Note from Table I that the only crucial parameters that were changed from the previous calculation are δt, μ, and a. Again, the three plots are of $u : x$, $I : x$, and $\rho : x$, respectively. In this case, ρ_{-}/ρ_{+} comes to the proper theoretical value of 1.21.

The technique is being applied to calculate wake flow through all Mach number variations from zero (incompressible flow with von Kármán vortex street) to values exceeding unity (supersonic flow with Prandtl–Meyer turning and wake-shock formation). These results will be reported elsewhere.

Acknowledgments

We are grateful to C. W. Hirt for discussions of the stability and accuracy derivation technique. This work was performed under the auspices of the United States Atomic Energy Commission.

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